

# Hamiltonian solutions of the 3-body problem in (2+1)-gravity

M. Ciafaloni

*Dipartimento di Fisica, Università di Firenze and INFN, Sezione di Firenze,  
50019 Sesto Fiorentino, Italy*

S. Munier

*Centre de Physique Théorique, École Polytechnique, CNRS, 91128 Palaiseau, France*

September 16, 2011

## Abstract

We present a full study of the 3-body problem in gravity in flat (2+1)-dimensional space-time, and in the nonrelativistic limit of small velocities. We provide an explicit form of the ADM Hamiltonian in a regular coordinate system and we set up all the ingredients for canonical quantization. We emphasize the rôle of a  $U(2)$  symmetry under which the Hamiltonian is invariant and which should generalize to a  $U(N-1)$  symmetry for  $N$  bodies. This symmetry seems to stem from a braid group structure in the operations of looping of particles around each other, and guarantees the single-valuedness of the Hamiltonian. Its rôle for the construction of single-valued energy eigenfunctions is also discussed.

## 1 Introduction

The gravitational problem with or without matter in 2+1 dimensions [1] has received considerable attention in the past years (for reviews, see Ref. [2, 3]) as a laboratory for a nonperturbative treatment of gravity at the classical and quantum level. Recently, much work has been devoted to topologically massive 2+1-gravity in AdS spacetimes (see e.g. [4] and references therein), but several aspects of the more conventional problem with matter in an open spacetime are still unsolved, and addressed here.

In fact, despite several efforts, an explicit quantum mechanical treatment of  $O(2,1)$  gravity with matter has been found only in the 2-body case, following the Deser-Jackiw-'t Hooft (DJH) classical solution [5] and its quantization [6, 7]. Subsequently, progress in the treatment of the classical  $N$ -body case with a regular metric has been achieved both in the first-order formalism [8, 9] and, more implicitly, in the canonical one [11] by using a York-type gauge. But no conclusive work on the solutions of the  $N$ -body quantum gravity problem is yet available for  $N \geq 3$ .

The purpose of the present paper is to perform further steps in the direction of a canonical quantum treatment of the  $N = 3$  case. By using the simplifying assumption of small velocities [10, 9], we are able to provide the (single-valued) Hamiltonian and a complete set of constants of motion in a fully explicit form and in a regular coordinate system. Canonical quantization is then, in principle, straightforward. However, we have not been able, so far, to implement the monodromy condition on the energy eigenfunctions, and thus to construct the canonical Hilbert space.

We use, throughout the paper, a regular coordinate system, such that the metric be single-valued everywhere, including in the neighbourhoods of (pointlike) external matter. On the other hand, it is known [1] that space-time is flat outside matter sources, so that one can use instead

Cartesian coordinates – characterized by various deficit angles – in which the conjugate momenta to the particles’ positions are constants of motion. The mapping from regular coordinates in a York-type gauge to Cartesian coordinates was constructed in Ref. [9] thus providing an explicit expression of the constants of motion both for  $N = 2$  and in the nonrelativistic  $N = 3$  case.

The canonical formalism was then set up in Ref. [11] by providing the form of the Hamiltonian for  $N = 2$  and its implicit definition for  $N \geq 3$ . Its general interpretation is that of the scale factor in the asymptotic Liouville field occurring in the ADM parametrization of the metric. The external masses act as sources of the Liouville field, together with the so-called apparent singularities [12] of the problem. Therefore, the asymptotic scale factor is a function of the particle masses and coordinates which can in principle be computed.

Our first task here is to provide the explicit form of the Hamiltonian for the 3-body case, on the basis of its definition in Ref. [11]. We are able to do that in the small-velocity limit, where we show that the Hamiltonian is simply related to a sum of squared moduli of two properly-defined relative momenta (called  $P_3$  and  $P_2$  in the paper), whose expressions in terms of regular particle coordinates and momenta are explicitly given in Sec. 3. Since such expressions carry branch cuts, there are nontrivial monodromy transformations for  $P_3$  and  $P_2$  when the particle positions (of particles #2 and #3, say) turn around each other. Nevertheless, the Hamiltonian is left invariant (and is thus single-valued) because it possesses a  $U(2)$  symmetry and, moreover, the pair  $(P_3, P_2)$  turns out to transform as a  $U(2)$  spinor.

The  $U(2)$  invariance of our problem has thus an important rôle in assessing the monodromy of the Hamiltonian, and the exchange symmetry of its equation of motion. We feel it should have a rôle in quantization as well, because it regulates the degeneracy of the wave functions and their monodromy properties.

We are thus able to provide explicit solutions of the classical Hamilton equations and to express them in terms of Cartesian coordinates and momenta. Such solutions agree with those found in Ref. [9] in the first-order formalism. Given such explicit understanding of the Hamiltonian structure, canonical quantization is in principle straightforward, and is here formulated by using a proper ordering prescription for the expression of the Hamiltonian in terms of regular coordinates and momenta. However, while the Hamiltonian is single valued, its eigenfunctions – characterized by a large degeneracy – are generically multiple-valued under the braiding of the 2 and 3 labels, for instance. We have some ideas on how to possibly construct monodromic eigenstates, but we have not found a successful procedure yet.

After summarizing previous work on the canonical formalism and describing how to calculate the Hamiltonian in the nonrelativistic limit in Sec. 2, we provide its explicit form for  $N = 3$  in Sec. 3, where we discuss its  $U(2)$  symmetry also. The equations of motion and their classical solutions are given in Sec. 4, the comparison with previous calculations in a different formalism is done in Sec. 5 while in Sec. 6, we sketch the quantization of the problem. We summarize our results and outline a few ideas which may prove useful in order to construct the quantum Hilbert space in the final section. Some technical details are discussed in the appendix.

## 2 Hamiltonian formulation and nonrelativistic limit

We want to describe the motion of  $N$  pointlike massive particles in a 2+1 dimensional open universe in a Hamiltonian formalism, in order to be able to perform canonical quantization. The complete formalism being available in several places in the literature [13, 11] (see also [8, 14]), we shall not enter the details of the derivation. We provide the essential formulae in the general case in the first subsection, and specialize to the nonrelativistic limit in the next one.

## 2.1 General Hamiltonian for the motion of $N$ pointlike massive particles

Let us start with the action in the Arnowitt-Deser-Misner (ADM) formulation [15, 16]. Parametrizing the generic line element  $ds$  in the standard form

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

the action reads

$$S = \int dt \left\{ \sum_n p_{ni} \dot{q}_n^i + \int d^2x (\Pi^{ij} \dot{h}_{ij} - N^i \mathcal{H}_i - N \mathcal{H}) \right\} + S_B, \quad (2)$$

where  $q_n$  is the position of particle number  $n$  ( $p_n$  is the conjugate momentum),  $\Pi$  is the canonical momentum conjugate to the spatial metric  $h$ , and

$$\begin{aligned} \mathcal{H}_i &= -2\sqrt{h} \nabla_j \frac{\Pi^j_i}{\sqrt{h}} - \sum_n \delta^2(x - q_n) p_{ni} \\ \mathcal{H} &= \frac{2\kappa^2}{\sqrt{h}} \left[ \Pi^i_j \Pi^j_i - (\Pi^i_i)^2 \right] - \frac{\sqrt{h} \mathcal{R}}{2\kappa^2} + \sum_n \delta^2(x - q_n) \sqrt{m_n^2 + h^{ij} p_{ni} p_{nj}}. \end{aligned} \quad (3)$$

$\mathcal{R}$  is the intrinsic curvature of the 2-spacelike slice and  $\nabla$  is the covariant derivative compatible with the 2-dimensional metric  $h_{ij}$ .  $\kappa^2$  is related to Newton's constant through  $\kappa^2 = 8\pi G$ , and has dimensions of an inverse mass in 2-dimensional space. It was set to  $\frac{1}{2}$  in Ref. [13, 11], and to 1 in Ref. [5, 8, 9]. We refer the reader to [17, 13] for a complete expression including the boundary terms  $S_B$ .

We can check that the variation of the action in Eq. (2) with respect to the metric components  $N$ ,  $N^i$ ,  $h_{ij}$  and to the momentum  $\Pi_{ij}$  yields the Einstein equations in a first-order form. In particular, one gets the equations  $\mathcal{H} = 0$  and  $\mathcal{H}_i = 0$  which are the so-called Hamiltonian and momentum constraints respectively, and which we shall analyze below. The Hamiltonian for the particles' motion is obtained after all variables have been expressed as a function of  $p_n$  and  $q_n$ .

We choose the York instantaneous gauge in which the intrinsic curvature  $K$  is zero everywhere, which has proved useful in this context. Introducing complex notation  $z \equiv x^1 + ix^2$ ,  $z_n \equiv q_n^1 + iq_n^2$ , and  $p_n \equiv (p_n^1 - ip_n^2)/2$ , we impose the further gauge-fixing conditions:  $h_{zz} = 0$  and  $h_{\bar{z}\bar{z}} = 0$ . The spatial line element then takes the conformally flat form

$$dl^2 = e^{2\sigma} |dz|^2 \quad (\text{or, equivalently, } h_{ij} = e^{2\sigma} \delta_{ij}). \quad (4)$$

This equation defines the field  $\sigma$ .

The momentum and Hamiltonian constraints can now be solved for  $\Pi$  and  $\sigma$ . The momentum constraint  $\mathcal{H}_i = 0$  determines the components of the energy-momentum tensor:

$$\Pi \equiv \Pi^{\bar{z}}_z = -\frac{1}{2\pi} \sum_n \frac{p_n}{z - z_n} \quad (5)$$

(up to an entire function of the  $z$  variable which has to be set to zero to ensure good asymptotic properties of the metric, see Ref. [13]). We will work in a center-of-mass frame in which  $\sum_n p_n = 0$ . In this case,  $\Pi$  is the ratio of two polynomials of respective degree  $N - 2$  and  $N$ . Let us denote by  $z_A$  the zeros of the polynomial in the numerator. They are functions of the canonical variables  $z_n$  and  $p_n$ . We write

$$\Pi = -\frac{D}{2\pi} \frac{\prod_A (z - z_A)}{\prod_n (z - z_n)}, \quad (6)$$

where the normalization  $D$  is the so-called dilation factor

$$D \equiv \sum_n z_n p_n, \quad (7)$$

whose imaginary part is half of the total angular momentum of the system of particles.

The Hamiltonian constraint  $\mathcal{H} = 0$  reduces to the nonlinear equation

$$\Delta(2\sigma) = -2|2\kappa^2\Pi|^2 e^{-2\sigma} - 4\pi \sum_n \mu_n \delta^2(z - z_n), \quad (8)$$

where  $\Delta \equiv \partial_x^2 + \partial_y^2 = 4\partial_z \partial_{\bar{z}}$  and where we have introduced the dimensionless masses  $\mu_n \equiv \kappa^2 m_n / 2\pi$ . Defining

$$e^{-2\tilde{\sigma}} \equiv 2|2\kappa^2\Pi|^2 e^{-2\sigma}, \quad (9)$$

the new field  $\tilde{\sigma}$  obeys a Liouville equation:

$$\Delta(2\tilde{\sigma}) = -e^{-2\tilde{\sigma}} - 4\pi \sum_n (\mu_n - 1) \delta^2(z - z_n) - 4\pi \sum_A \delta^2(z - z_A). \quad (10)$$

There are  $N$  sources located at the particles' positions  $z_n$ . The remaining sources  $z_A$  are so-called apparent singularities: They stem from the zeros of the momentum tensor  $\Pi$ . Their number is  $N - 2$  in the center-of-mass frame.

The Euler characteristic of the 2-surface is an important parameter. It reads

$$2\pi\mu = \frac{1}{2} \int d^2z \sqrt{h} \mathcal{R} = -\frac{1}{2} \int d^2z \Delta(2\sigma) = \lim_{r \rightarrow \infty} -\frac{1}{2} \oint_r \vec{\nabla}(2\sigma) \cdot d\vec{n}. \quad (11)$$

The notation for the integration element is  $d^2z \equiv dxdy = dzd\bar{z}/(2i)$ . The last equality, obtained from the Gauss law on a circle of radius  $r$  with exterior normal  $\vec{n}$ , provides the asymptotic behavior of  $\sigma$  in the form

$$e^{2\sigma} \underset{|z| \gg |\lambda|}{\sim} \left| \frac{z}{\lambda} \right|^{-2\mu}, \quad (12)$$

where  $\lambda$  is a distance scale.

The parameter  $\mu$  is a constant of motion and is interpreted as the (rescaled) total mass of the universe. If all particles are static, there is no interaction energy, since in 2+1 dimensional gravity, spacetime is flat outside the sources, and thus there are no local interactions. Hence in the static case, the total mass of the universe is the sum of the masses of all particles:  $\mu = \sum_n \mu_n$ .

It was found in Ref. [11] that the physical Hamiltonian is related to the logarithm of the scale  $\lambda$  of the spatial distances, namely

$$H = \frac{\mu}{2\kappa^2} \ln |\lambda|^2 \quad (13)$$

up to a constant term, where  $\lambda$  is a function of the canonical variables which are the particle momenta  $p_n$  and positions  $z_n$ . To give sense to Eq. (13), one has to make  $\lambda$  dimensionless by dividing out a basic length scale (which does not appear to be fundamental, but rather related to the initial conditions.) In the following, all length variables ( $\lambda, z, z_n \dots$ ) will be considered dimensionless.

The Hamiltonian for the 2-body problem can be computed exactly from Eq. (13). However, for larger values of  $N$ , the exact resolution of the Hamiltonian constraint (8) is already a formidable task [18]. Therefore, in this paper, we shall stick to the nonrelativistic approximation which was proposed in Ref. [9], and which we shall now derive in the Hamiltonian formulation.

## 2.2 Nonrelativistic limit

To characterize the quasi-static approximation, we have at our disposal the  $N + 1$  parameters  $\mu$  and  $\mu_n$ . When the particles are at rest, the difference between the total mass of the universe and the sum of the masses of the particles

$$\varepsilon \equiv \mu - \sum_n \mu_n \quad (14)$$

is zero. So in the following, we shall always restrict ourselves to the lowest order in  $\varepsilon$ .

Let us start from the definition of the Euler characteristic (11). Inserting Eq. (8) in Eq. (11), we obtain the following relation for  $\tilde{\sigma}$  defined in Eq. (9):

$$\frac{1}{2\pi} \int d^2 z e^{-2\tilde{\sigma}} = \varepsilon. \quad (15)$$

An analysis of Eq. (10) shows that on the particle singularities, the term  $e^{-2\tilde{\sigma}}$  has to behave like  $|z - z_n|^{2(\mu_n-1)}$ . (This is consistent with the fact that  $e^{-2\sigma}$  vanish for  $z = z_n$ , as long as the masses  $\mu_n$  are positive). In the right-hand side of Eq. (10), the  $\delta$ -functions then dominate at the particle singularities. Choosing a regularization, we see that from Eq. (15), the term  $e^{-2\tilde{\sigma}}$  is of order  $\varepsilon$ . The scale  $\varepsilon$  disappears when one takes the Laplacian of its logarithm as in the left-hand side. So setting  $\varepsilon$  to zero, we see that the nonlinear term drops out and the Liouville equation for  $\tilde{\sigma}$  boils down to a Poisson equation. The solution of the latter reads

$$e^{-2\tilde{\sigma}(z)} = |K|^2 \prod_{n=1}^N |z - z_n|^{2\mu_n-2} \prod_{A=1}^{N-2} |z - z_A|^2, \quad (16)$$

where  $K$  is a complex number independent of  $z$ . We note that this solution is consistent with the one found in Ref. [9]. There, the analysis was performed in a first-order formalism, using the scale of the Cartesian particle velocities as a small parameter. We shall make the comparison sharper later on.

To obtain the Hamiltonian, we only need the large-distance behavior of  $e^{-2\sigma}$  which can be deduced from the behavior of  $e^{-2\tilde{\sigma}}$  and of  $\Pi$  using Eq. (9). First, from Eq. (16),

$$e^{-2\tilde{\sigma}} \underset{|z| \rightarrow \infty}{\sim} |K|^2 |z|^{\sum \mu_n - 2}. \quad (17)$$

Second, from Eq. (6), the momentum  $\Pi$  behaves like  $D/z^2$ , where  $D$  is defined in Eq. (7). Identifying the asymptotic behavior of  $e^{-2\sigma}$  just found with Eq. (12) at the lowest order in  $\varepsilon$ , we get an expression for  $|\lambda|^2$  from which, with the help of Eq. (13), we deduce the following formula for the Hamiltonian:

$$H = \frac{1}{2\kappa^2} \ln \frac{|2\kappa^2 D|^2}{|K|^2}, \quad (18)$$

up to an irrelevant constant. The parameter  $|K|^2$  can be determined as a function of the canonical variables and of the constant  $\varepsilon$  using Eq. (15). So our task is now to integrate Eq. (16) over the whole complex plane. An appropriate change of variable enables us to cast the integral in the form

$$\varepsilon = \frac{1}{4\pi} |K|^2 |z_{21}|^{2\mu-2} \int d^2 \xi |\xi|^{2\mu_1-2} |1 - \xi|^{2\mu_2-2} |\zeta_3 - \xi|^{2\mu_3-2} \dots |\zeta_N - \xi|^{2\mu_N-2} \prod_A |\zeta_A - \xi|^2 \quad (19)$$

where we have defined the new variable  $\zeta_{n/A} \equiv (z_{n/A} - z_1)/(z_2 - z_1)$ ,  $z_{ij} = z_i - z_j$  and  $\mu = \sum_{n=1}^N \mu_n$  consistently with the quasi-static approximation.

It is easy to check that these formulae allow to recover, up to an additive constant, the Hamiltonian of the 2-body problem, which was written down by several groups [5, 6, 7, 8, 9, 11]:

$$H = \frac{1}{2\kappa^2} (\ln |2\kappa^2 p|^2 + \mu \ln |z|^2), \quad (20)$$

where  $p$  is the momentum  $p_2$  of particle #2 (keeping in mind that  $p_1 + p_2 = 0$ ),  $z \equiv z_{21} = z_2 - z_1$ .

### 3 Explicit Hamiltonian for the three-body problem

So far, we have obtained a general formula for the Hamiltonian in the nonrelativistic limit as a function of the canonical positions and momenta of the particles, described in a regular coordinate system by the variables  $z_n$  and  $p_n$  respectively. It is given in Eq. (18), with  $D$  defined in Eq. (7) and  $|K|^2$  obtained from the evaluation of the integral in (19). In the present and the following sections, we shall specialize to  $N = 3$  bodies. In this case, the nonrelativistic Hamiltonian may still be expressed in terms of known functions and, therefore, may be studied completely.

Interesting new features appear with respect to the two-body case. In particular, the Hamiltonian possesses a  $U(2)$  invariance, related to a braid group structure of the particle exchanges and loopings.

#### 3.1 Explicit calculation

Let us first gather the ingredients for the computation of the Hamiltonian (18), namely the expressions of  $|K|^2$  and of  $D$ .

We specialize Eq. (19) to the case of 3 bodies. The rescaled position variables of particle #3 and of the apparent singularity respectively read

$$\zeta \equiv \zeta_3 = \frac{z_{31}}{z_{21}}, \quad \zeta_A = \frac{(p_2 + p_3)\zeta}{p_2 + p_3\zeta}. \quad (21)$$

We choose the origin of the frame at the position of particle #1 in such a way that  $z_1 = 0$ . Then Eq. (19) reads

$$\varepsilon = \frac{1}{4\pi} |K|^2 |z_2|^{2\mu-2} \int d^2\xi |\xi|^{2\mu_1-2} |1 - \xi|^{2\mu_2-2} |\zeta - \xi|^{2\mu_3-2} |\zeta_A - \xi|^2, \quad (22)$$

where  $\mu = \mu_1 + \mu_2 + \mu_3$  in the right-hand side of this equation. The dilation operator defined in Eq. (7) is

$$D = z_2(p_2 + p_3\zeta). \quad (23)$$

We now compute the integral that defines  $\varepsilon$  (Eq. (22)). To this aim, we first expand

$$|\xi - \zeta_A|^2 = |\xi - \zeta|^2 + (\xi - \zeta)(\bar{\zeta} - \bar{\zeta}_A) + (\bar{\xi} - \bar{\zeta})(\zeta - \zeta_A) + |\zeta - \zeta_A|^2 \quad (24)$$

in order to be able to cast  $\varepsilon$  in the form of a sum of standard integrals:

$$\varepsilon = |K|^2 |z_2|^{2\mu-2} [J_{11} + (\bar{\zeta}_A - \bar{\zeta})J_{10} + (\zeta_A - \zeta)J_{01} + |\zeta_A - \zeta|^2 J_{00}], \quad (25)$$

where

$$J_{\delta\bar{\delta}} = \int d^2\xi |\xi|^{2(\mu_1-1)} |1 - \xi|^{2(\mu_2-1)} (\zeta - \xi)^{\mu_3-1+\delta} (\bar{\zeta} - \bar{\xi})^{\mu_3-1+\bar{\delta}}. \quad (26)$$

This kind of integrals appears in conformal field theory and were computed e.g. in Refs. [20, 21]. The  $J$ -functions can be expressed with the help of hypergeometric functions:

$$\begin{aligned}
J_{\delta\tilde{\delta}} = & \zeta^{\mu_1+\mu_3+\delta-1} \bar{\zeta}^{\mu_1+\mu_3+\tilde{\delta}-1} B_{\mu_1, \mu_3+\delta} F(\mu_1, 1-\mu_2, \mu_1+\mu_3+\delta, \zeta) \\
& \times B_{\mu_1, \mu_3+\tilde{\delta}} F(\mu_1, 1-\mu_2, \mu_1+\mu_3+\tilde{\delta}, \bar{\zeta}) \frac{s_1 s_3}{s_{13}} \\
& + (-1)^{\delta-\tilde{\delta}} B_{\mu_2, \mu_1+\mu_3+\delta-1} F(1-\mu_3-\delta, 2-\mu_1-\mu_2-\mu_3-\delta, 2-\mu_1-\mu_3-\delta, \zeta) \\
& \times B_{\mu_2, \mu_1+\mu_3+\tilde{\delta}-1} F(1-\mu_3-\tilde{\delta}, 2-\mu_1-\mu_2-\mu_3-\tilde{\delta}, 2-\mu_1-\mu_3-\tilde{\delta}, \bar{\zeta}) \frac{s_2 s_{13}}{s_{123}}, \quad (27)
\end{aligned}$$

where  $s_i = \sin \pi \mu_i$ ,  $s_{ij\dots} = \sin \pi(\mu_i + \mu_j + \dots)$  and  $B_{\alpha, \beta}$  is the standard Euler beta function. Note that since we are working at the lowest order in  $\varepsilon$ , we may always replace  $\sum \mu_i$  by the total mass  $\mu$  in each term that appears in the r.h.s. of Eq. (25). We define the four following functions:

$$\begin{aligned}
f_{3a} &= B_{\mu_1, \mu_3} \zeta^{\mu_1+\mu_3-1} F(\mu_1, 1-\mu_2, \mu_1+\mu_3, \zeta), & f_{3b} &= f_{3a}|_{\mu_3 \rightarrow 1+\mu_3}, \\
f_{2a} &= B_{\mu_1+\mu_3-1, \mu_2} F(1-\mu_3, 2-\mu, 2-\mu_1-\mu_3, \zeta), & f_{2b} &= f_{2a}|_{\mu_3 \rightarrow 1+\mu_3}.
\end{aligned} \quad (28)$$

The functions  $f_{3a}$  and  $f_{2a}$  of the variable  $\zeta$  are two independent solutions of the hypergeometric equation

$$\zeta(1-\zeta)f''_{ia} + [2-\mu_1-\mu_3-(3-\mu_3-\mu)\zeta]f'_{ia} - (2-\mu)(1-\mu_3)f_{ia} = 0 \quad (29)$$

and, similarly,  $f_{3b}$  and  $f_{2b}$  solve the hypergeometric equation obtained from the previous one after having performed the shift  $\mu_3 \rightarrow \mu_3 + 1$ .

After some algebra, involving in particular the trigonometric identity

$$\sin \pi \alpha \sin \pi(\alpha + \beta + \gamma) = \sin \pi(\alpha + \beta) \sin \pi(\alpha + \gamma) - \sin \pi \beta \sin \pi \gamma \quad (30)$$

with  $\alpha = \mu_1$ ,  $\beta = \mu_2$  and  $\gamma = \mu_3$ , we arrive at an expression of  $\varepsilon/|K|^2$  in terms of the  $f$ 's:

$$\frac{4\varepsilon}{|K|^2} = |z_2|^{2\mu-2} [N_3|(\zeta_A - \zeta)f_{3a} + f_{3b}|^2 + N_2|(\zeta_A - \zeta)f_{2a} - f_{2b}|^2], \quad (31)$$

where

$$N_2 \equiv \frac{s_2 s_{13}}{\pi s_{123}}, \quad N_3 \equiv \frac{s_1 s_3}{\pi s_{13}}. \quad (32)$$

The Hamiltonian (18) then reads

$$H = \frac{1}{2\kappa^2} \left\{ \ln \frac{|2\kappa^2 D|^2}{4\varepsilon} + (\mu-1) \ln |z_2|^2 + \ln [N_3|(\zeta_A - \zeta)f_{3a} + f_{3b}|^2 + N_2|(\zeta_A - \zeta)f_{2a} - f_{2b}|^2] \right\}. \quad (33)$$

Introducing some more notations, whose physical interpretation will be given below, the Hamiltonian (33) may be rewritten in a compact form

$$H = \frac{1}{2\kappa^2} \ln \frac{(2\kappa^2)^2 (|P_2|^2 + |P_3|^2)}{4\varepsilon}, \quad (34)$$

where we have defined

$$P_2 \equiv D z_2^{\mu-1} \sqrt{N_2} [f_{2b} - (\zeta_A - \zeta)f_{2a}] \quad \text{and} \quad P_3 \equiv D z_2^{\mu-1} \sqrt{N_3} [f_{3b} + (\zeta_A - \zeta)f_{3a}] \quad (35)$$

or, written in terms of the canonical variables  $z_2, \zeta$  and  $p_2, p_3$  with the help of Eqs. (21) and (23),

$$\begin{aligned}
P_2(z_2, \zeta, p_2, p_3) &= z_2^\mu F_2(\zeta, p_2, p_3), & F_2 &= \alpha(\zeta)p_2 + \beta(\zeta)p_3 \\
P_3(z_2, \zeta, p_2, p_3) &= z_2^\mu F_3(\zeta, p_2, p_3), & F_3 &= \gamma(\zeta)p_2 + \delta(\zeta)p_3,
\end{aligned} \quad (36)$$

the coefficients  $\alpha, \beta, \gamma, \delta$  in front of the momenta being defined as

$$\begin{aligned}\alpha &= \sqrt{N_2} f_{2b}, \quad \beta = \sqrt{N_2} \zeta (f_{2b} - (1 - \zeta) f_{2a}), \\ \gamma &= \sqrt{N_3} f_{3b}, \quad \delta = \sqrt{N_3} \zeta (f_{3b} + (1 - \zeta) f_{3a}).\end{aligned}\tag{37}$$

This set of notations will help to write the calculation of Sec. 4 in a simpler way.

### 3.2 Interpretation of $P_3$ and $P_2$ in the small mass limit

Before embarking with the study of the Hamiltonian (34), we wish to try and give an interpretation of its expression. It is quite straightforward in the small mass limit  $\mu_n \ll 1$  for  $n = 1, 2, 3$  in which gravity effects vanish. Let us write the expression of  $P_2$  and  $P_3$  in Eq. (36) in this limit. We start with the  $f$ 's defined in Eq. (28): At the lowest order in all the  $\mu_n$ , they boil down to

$$f_{2a} \simeq \frac{\mu}{\mu_{13}\mu_2} \frac{1}{1 - \zeta}, \quad f_{2b} \simeq \frac{\mu}{\mu_{13}\mu_2}, \quad f_{3a} \simeq \frac{\mu_{13}}{\mu_1\mu_3} \frac{1}{\zeta} + \frac{1}{\mu_3} \frac{1}{1 - \zeta}, \quad f_{3b} \simeq \frac{1}{\mu_1},\tag{38}$$

and it follows that

$$P_2 \simeq \frac{1}{\sqrt{N_2}} \frac{\mu_2\mu_{13}}{\mu} \left( \frac{p_2}{\mu_2} - \frac{p_1 + p_3}{\mu_{13}} \right), \quad P_3 \simeq \frac{1}{\sqrt{N_3}} \frac{\mu_1\mu_3}{\mu_{13}} \left( \frac{p_3}{\mu_3} - \frac{p_1}{\mu_1} \right).\tag{39}$$

Thus we see that  $P_3$  is the relative momentum of particles #1 and #3 with the normalization  $\sqrt{N_3} \simeq \sqrt{\mu_1\mu_3/\mu_{13}}$ , and  $P_2$  is the relative momentum of particle #2 with respect to the (13) subsystem, with the normalization  $\sqrt{N_2} \simeq \sqrt{\mu_2\mu_{13}/\mu}$ . In the absence of interaction, the nonrelativistic kinetic energy of 3 pointlike particles reads

$$E = 2\kappa^2 \left( \frac{\vec{p}_1^2}{2\mu_1} + \frac{\vec{p}_2^2}{2\mu_2} + \frac{\vec{p}_3^2}{2\mu_3} \right) = 4\kappa^2 \left( \frac{|p_1|^2}{\mu_1} + \frac{|p_2|^2}{\mu_2} + \frac{|p_3|^2}{\mu_3} \right).\tag{40}$$

With the help of the expressions for  $P_2$  and  $P_3$  in the small mass limit given by Eq. (39), it can be rewritten as

$$E = 4\kappa^2 (|P_2|^2 + |P_3|^2).\tag{41}$$

Note that this kinetic energy does not coincide with the Hamiltonian in Eq. (34), which reads

$$H = \frac{1}{2\kappa^2} \ln \frac{\kappa^2 E}{4\varepsilon}.\tag{42}$$

This is actually related to the time gauge which was chosen, and is better discussed in Sec. 4.3.

### 3.3 Monodromy properties, $U(2)$ symmetry and braids

We want to check that the Hamiltonian (33) is well-defined. It is necessary that it be single-valued, i.e. invariant under the loopings of  $\zeta$  around the branch points at 0, 1 and  $\infty$ , and invariant under the exchange of the labels of the particles.

#### 3.3.1 Monodromies and $U(2)$ symmetry

Let us introduce the two objects

$$\sigma_b = \begin{pmatrix} \sqrt{N_3} f_{3b} \\ \sqrt{N_2} f_{2b} \end{pmatrix} \quad \text{and} \quad \sigma_a = \begin{pmatrix} \sqrt{N_3} f_{3a} \\ -\sqrt{N_2} f_{2a} \end{pmatrix}\tag{43}$$

in such a way that the momenta  $(P_3, P_2)$  defined in Eq. (35) can be conveniently rewritten as

$$P \equiv \begin{pmatrix} P_3 \\ P_2 \end{pmatrix} = Dz_2^{\mu-1} [\sigma_b + (\zeta_A - \zeta) \sigma_a].\tag{44}$$



We compute the monodromy matrices  $M_{32}$  and  $M_{31}$  which correspond to the loopings of particle #3 around particle #2, and #3 around #1 respectively. These transformations amount to substituting  $\zeta - 1 \rightarrow e^{2i\pi}(\zeta - 1)$  and  $\zeta \rightarrow e^{2i\pi}\zeta$ . We find that  $\sigma_a$  and  $\sigma_b$  transform according to the matrix (see Appendix A for the details)

$$M_{31} = e^{i\pi(\mu_1+\mu_3)} \begin{pmatrix} e^{i\pi(\mu_1+\mu_3)} & 0 \\ 0 & e^{-i\pi(\mu_1+\mu_3)} \end{pmatrix} \quad (45)$$

for the looping of particle #3 around particle #1. As for the looping of particle #3 around particle #2, the corresponding transformation reads

$$M_{32} = e^{i\pi(\mu_2+\mu_3)} \begin{pmatrix} a_{32} & b_{32} \\ -\bar{b}_{32} & \bar{a}_{32} \end{pmatrix} \quad \text{with} \quad \begin{cases} a_{32} = \cos \pi(\mu_2 + \mu_3) + i \sin \pi(\mu_2 + \mu_3) \cos \alpha_{32}, \\ b_{32} = i \sin \pi(\mu_2 + \mu_3) \sin \alpha_{32}, \end{cases} \quad (46)$$

and where the angle  $\alpha_{32}$  has been defined as

$$\cos \alpha_{32} = \frac{s_1 s_2 - s_3 s_{123}}{s_{13} s_{23}}, \quad \sin \alpha_{32} = -\frac{2\sqrt{s_1 s_2 s_3 s_{123}}}{s_{13} s_{23}}. \quad (47)$$

The matrices  $\tilde{M}_{31} = M_{31}e^{-i\pi(\mu_1+\mu_3)}$  and  $\tilde{M}_{32} = M_{32}e^{-i\pi(\mu_2+\mu_3)}$  are  $SU(2)$  transformations, so that the monodromy group generated by  $M_{31}$  and  $M_{32}$  is actually a subgroup of  $U(2)$ .

The object  $(\zeta_A - \zeta)\sigma_a$  also transforms according to  $M_{31}$  and  $M_{32}$ , and the same holds for the sum  $(\zeta_A - \zeta)\sigma_a + \sigma_b$ . This implies that the norm of this  $SU(2)$  spinor, which reads

$$N_3 |(\zeta_A - \zeta)f_{3a} + f_{3b}|^2 + N_2 |(\zeta_A - \zeta)f_{2a} - f_{2b}|^2 \quad (48)$$

is stable under the monodromy transformations. Consequently, the Hamiltonian (33) is single valued under all possible loopings of particle #3 around the two other particles.

We have not considered the looping of particle #2 around particle #1. The corresponding transformation of the spinors can be computed: The best is to exchange #3 and #2 in such a way that the looping of particle #2 around #1 be the substitution  $\zeta \rightarrow e^{2i\pi}\zeta$ , and eventually, to go back to the initial coordinates by applying the inverse exchange transformation. Therefore, we postpone this discussion after we have studied how the momenta change when the particles are relabeled.

### 3.3.2 Relabeling symmetry

Exchanging the labels of the particles yields nontrivial transformations of the spinor  $(P_3, P_2)$ . We now check that the Hamiltonian is invariant under these transformations.

We first perform the exchange  $\mu_1 \leftrightarrow \mu_3$  and  $(z_1, p_1) \leftrightarrow (z_3, p_3)$  in the expression of the  $P$ 's in Eq. (36). Since the details are lengthy, we defer the full calculation to Appendix A. We find that the transformation of the spinor  $(P_3, P_2)$  can be written as the multiplication by the diagonal matrix

$$\tau_{31}^{\mu_3\mu_1} = e^{i\pi(\mu_1+\mu_3)/2} \begin{pmatrix} -e^{i\pi(\mu_1+\mu_3)/2} & 0 \\ 0 & e^{-i\pi(\mu_1+\mu_3)/2} \end{pmatrix}. \quad (49)$$

In the same way, the exchange of the labels of particles #3 and #2 is represented by the multiplication by the matrix

$$\tau_{32}^{\mu_3\mu_2(\mu_1)} \equiv \frac{e^{i\pi(\mu_2+\mu_3)/2}}{\sqrt{s_{12}s_{13}}} \begin{pmatrix} e^{-i\pi(\mu_1+\mu_2/2+\mu_3/2)}\sqrt{s_3 s_2} & e^{i\pi(\mu_3-\mu_2)/2}\sqrt{s_1 s_{123}} \\ e^{i\pi(\mu_2-\mu_3)/2}\sqrt{s_1 s_{123}} & -e^{i\pi(\mu_1+\mu_2/2+\mu_3/2)}\sqrt{s_2 s_3} \end{pmatrix}. \quad (50)$$

We have written the two matrices as the product of a  $U(1)$  phase and a  $U(2)$  matrix of determinant  $-1$ . (This is clear in the former case, and can be easily checked using the trigonometric

identity (30) in the latter case.) It is then obvious that the spinorial norm  $|P_2|^2 + |P_3|^2$  is left invariant by these transformations, and since  $\varepsilon$  is trivially invariant, the same is true for the full Hamiltonian (34).

We check that the monodromy transformations are the “squared” of the relabeling transformations. More precisely,

$$\tau_{31}^{\mu_1\mu_3} \tau_{31}^{\mu_3\mu_1} = M_{31} \quad \text{and} \quad \tau_{32}^{\mu_2\mu_3(\mu_1)} \tau_{32}^{\mu_3\mu_2(\mu_1)} = M_{32}. \quad (51)$$

We are now in a position to easily get the last monodromy transformation which we left uncomputed in the last subsection, namely the action of the looping of particle #2 around particle #1 on the spinor  $(P_3, P_2)$  that we shall denote by  $M_{21}$ . We write

$$M_{21} = (\tau_{32}^{-1})^{\mu_2\mu_3(\mu_1)} \tau_{31}^{\mu_1\mu_2} \tau_{31}^{\mu_2\mu_1} \tau_{32}^{\mu_3\mu_2(\mu_1)}. \quad (52)$$

The result of the matrix multiplication may be written in the same form as Eq. (46), except for the values of the  $U(1)$  phase and of the  $SU(2)$  angles:

$$M_{21} = e^{i\pi(\mu_1+\mu_2)} \begin{pmatrix} a_{21} & b_{21} \\ -\bar{b}_{21} & \bar{a}_{21} \end{pmatrix} \quad \text{with} \quad \begin{cases} a_{21} = \cos \pi(\mu_1 + \mu_2) + i \sin \pi(\mu_1 + \mu_2) \cos \alpha_{21}, \\ b_{21} = -ie^{i\pi(\mu_1+\mu_3)} \sin \pi(\mu_1 + \mu_2) \sin \alpha_{21}, \end{cases} \quad (53)$$

where

$$\cos \alpha_{21} = \frac{s_2 s_3 - s_1 s_{123}}{s_{12} s_{13}}, \quad \sin \alpha_{21} = -\frac{2\sqrt{s_1 s_2 s_3 s_{123}}}{s_{12} s_{13}}. \quad (54)$$

The monodromy  $M_{21}$  is nothing but the exchange of particles #2 and #1 repeated twice, which then has the following matrix representation:

$$\tau_{21}^{\mu_2\mu_1} \equiv (\tau_{32}^{-1})^{\mu_3\mu_2(\mu_1)} \tau_{31}^{\mu_2\mu_1} \tau_{32}^{\mu_3\mu_2(\mu_1)}, \quad (55)$$

and whose explicit expression reads

$$\tau_{21}^{\mu_2\mu_1} = \frac{e^{i\pi\mu_2}}{\sqrt{s_{13}s_{23}}} \begin{pmatrix} e^{-i\pi(\mu_1+\mu_3)} \sqrt{s_1 s_2} & -e^{i\pi(\mu_1+\mu_3)} \sqrt{s_3 s_{123}} \\ -e^{-i\pi(\mu_1+\mu_3)} \sqrt{s_3 s_{123}} & -e^{i\pi(\mu_1+\mu_3)} \sqrt{s_1 s_2} \end{pmatrix}. \quad (56)$$

### 3.3.3 Braids

There are interesting relations between the transformations (45),(46),(53),(49),(50),(56) which seem to point to some underlying braid group structure [22]. (It was anticipated by 't Hooft in Ref. [6] that braids and/or knots would play a rôle in this problem). We mention here the correspondence between the monodromy group and the braid group since it could be important for quantization, but without entering the details: We shall refer the interested reader to the litterature for an introduction to braids.

Let us consider 3 strings, each of them being attached to one of the particles, and which carry the mass of the latter. We assign the relabeling matrices  $\tau$  to the crossings of two out of the three strands. For each crossing, there are two possibilities, depending on which one of the particles passes above. We choose the convention that the matrix  $\tau_{32}^{\mu_i\mu_j(\mu_k)}$  moves the strand from the middle of the braid associated to particle #i of mass  $\mu_i$  above the strand initially at the bottom associated to particle #j. The matrix  $\tau_{31}^{\mu_i\mu_k}$  takes the strand at the top above the strand initially in the middle. The opposite orderings are represented by the inverses of the  $\tau$  matrices, namely

$$(\tau_{32}^{-1})^{\mu_i\mu_j(\mu_k)} \equiv (\tau_{32}^{\mu_j\mu_i(\mu_k)})^\dagger \quad \text{and} \quad (\tau_{31}^{-1})^{\mu_i\mu_k} \equiv (\tau_{31}^{\mu_k\mu_i})^\dagger. \quad (57)$$

Note that one has to keep proper track of the particle masses along the strands.

Now we check by an explicit calculation that the following relation hold:

$$\tau_{31}^{\mu_3\mu_2}\tau_{32}^{\mu_1\mu_2(\mu_3)}\tau_{31}^{\mu_3\mu_1} = \frac{e^{i\pi\mu}}{\sqrt{s_{13}s_{23}}} \begin{pmatrix} \sqrt{s_1s_2} & -\sqrt{s_3s_{123}} \\ -\sqrt{s_3s_{123}} & -\sqrt{s_1s_2} \end{pmatrix} = \tau_{32}^{\mu_1\mu_3(\mu_2)}\tau_{31}^{\mu_2\mu_1}\tau_{32}^{\mu_3\mu_2(\mu_1)}. \quad (58)$$

This identity is reminiscent of the defining property of the  $B_3$  group of braids on three strands, see Ref. [22].

Given the relation between monodromy and relabeling transformations, it is not difficult to check that the group generated multiplicatively by the matrices  $M_{31}$ ,  $M_{32}$  and  $M_{21}$  is homomorphic to the pure braid group on three strings. The latter is the subgroup of the braid group  $B_3$  which preserves the ordering of the strands. We check by explicit matrix multiplication that the two following defining relations of the pure braid group (see the corresponding equation in Ref. [22], which differs only by an appropriate relabeling) are identically verified:

$$M_{31}M_{32}M_{31}^{-1} = M_{21}^{-1}M_{32}M_{21}, \quad M_{31}M_{21}M_{31}^{-1} = M_{21}^{-1}M_{32}^{-1}M_{21}M_{32}M_{21}. \quad (59)$$

## 4 Equations of motion and conservation laws

So far, we have derived and studied the Hamiltonian which describes the evolution of three pointlike particles (see Eq. (34)). We are now going to investigate deeper the dynamics of the system. From the Hamilton-Jacobi equations, we will be able to compute the time evolution of the dilation factor (i.e. also of the total angular momentum) (Sec. 4.1). We will show that the momenta  $P_3$  and  $P_2$  in terms of which the Hamiltonian is written are conserved (Sec. 4.2), and we will define the positions  $Z_3$  and  $Z_2$  canonically conjugate to  $P_3$  and  $P_2$  respectively (Sec. 4.3).

In order to avoid to have to carry along  $2\kappa^2$  factors, let us take the momenta dimensionless by setting  $2\kappa^2 \equiv 1$ .

### 4.1 Hamilton-Jacobi equations and time evolution of the dilation factor

In our definition of complex positions and momenta (see Sec. 2), the Hamilton-Jacobi equations for the time evolution of the coordinates read

$$\dot{z}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{z}_3 = \frac{\partial H}{\partial p_3}, \quad (60)$$

that is, from Eqs. (34) and (36):

$$\dot{z}_2 = z_2^\mu \frac{\alpha \bar{P}_2 + \gamma \bar{P}_3}{|P_2|^2 + |P_3|^2}, \quad \dot{z}_3 = z_2^\mu \frac{\beta \bar{P}_2 + \delta \bar{P}_3}{|P_2|^2 + |P_3|^2}. \quad (61)$$

Comparing to the definitions (36), one immediately sees that

$$p_2 \dot{z}_2 + p_3 \dot{z}_3 = 1. \quad (62)$$

Similarly, the Hamilton-Jacobi equations for the evolution of the momenta read

$$\dot{p}_2 = -\frac{\partial H}{\partial z_2}, \quad \dot{p}_3 = -\frac{\partial H}{\partial z_3}. \quad (63)$$

Replacing  $H$  by its expression (34), one gets, after the further replacements of  $P_2$  and  $P_3$  by Eq. (36):

$$\begin{aligned} \dot{p}_2 &= \frac{\mu}{z_2} - \frac{z_2^\mu}{|P_2|^2 + |P_3|^2} \left( \frac{\partial F_2}{\partial z_2} \bar{P}_2 + \frac{\partial F_3}{\partial z_2} \bar{P}_3 \right) \\ \dot{p}_3 &= -\frac{z_2^\mu}{|P_2|^2 + |P_3|^2} \left( \frac{\partial F_2}{\partial z_3} \bar{P}_2 + \frac{\partial F_3}{\partial z_3} \bar{P}_3 \right). \end{aligned} \quad (64)$$

Since  $F_2$  and  $F_3$  only depend on  $\zeta$ , their derivatives with respect to  $z_2$  and  $z_3$  read

$$\frac{\partial}{\partial z_2} = -\frac{\zeta}{z_2} \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial z_3} = \frac{1}{z_2} \frac{\partial}{\partial \zeta}. \quad (65)$$

We then see that the following conservation law is satisfied:

$$\dot{p}_2 z_2 + \dot{p}_3 z_3 = -\mu. \quad (66)$$

Combining Eq. (62) and (66), we find that the dilation factor  $D$  defined in Eq. (23) has a linear evolution with the time  $t$ :

$$\dot{D} = 1 - \mu. \quad (67)$$

## 4.2 Cartesian momenta

We are going to prove that  $P_2$  and  $P_3$  are constants of motion. To this aim, we need to compute the Poisson brackets of the Hamiltonian with the momenta:

$$\dot{P}_2 = -\{H, P_2\} = \sum_{i=2,3} \left( -\frac{\partial H}{\partial z_i} \frac{\partial P_2}{\partial p_i} + \frac{\partial H}{\partial p_i} \frac{\partial P_2}{\partial z_i} \right) \quad (68)$$

and similarly  $\dot{P}_3 = -\{H, P_3\}$ . We easily find

$$\dot{P}_2 = \frac{\{P_2, P_3\} \bar{P}_3}{|P_2|^2 + |P_3|^2}, \quad \dot{P}_3 = \frac{\{P_3, P_2\} \bar{P}_2}{|P_2|^2 + |P_3|^2}. \quad (69)$$

We are going to show that the Poisson bracket  $\{P_2, P_3\}$  vanishes. Replacing  $P_2$  and  $P_3$  by their expressions (36), we get

$$\begin{aligned} \{P_2, P_3\} = z_2^{2\mu-1} \Big\{ & [\alpha'(\delta - \gamma\zeta) - \gamma'(\beta - \alpha\zeta)] p_2 \\ & + [\beta'(\delta - \gamma\zeta) - \delta'(\beta - \alpha\zeta) - \mu(\alpha\delta - \beta\gamma)] p_3 \Big\}. \end{aligned} \quad (70)$$

From the definitions of  $\alpha, \beta, \gamma, \delta$  in Eq. (37), the following identities hold:

$$\begin{aligned} \beta - \alpha\zeta &= -\sqrt{N_2}\zeta(1-\zeta)f_{2a} \\ \delta - \gamma\zeta &= \sqrt{N_3}\zeta(1-\zeta)f_{3a} \\ \alpha\delta - \beta\gamma &= \sqrt{N_2 N_3}\zeta(1-\zeta)(f_{2a}f_{3b} + f_{2b}f_{3a}). \end{aligned} \quad (71)$$

The derivatives of  $\alpha, \beta, \gamma, \delta$  can be expressed as a function of the  $f$ 's and the second derivatives of  $f_{2b}$  and  $f_{3b}$  by using

$$f'_{2b} = -\mu_3 f_{2a}, \quad f'_{3b} = \mu_3 f_{3a} \quad (72)$$

which are consequences of standard identities between hypergeometric functions.

We insert Eqs. (71),(72) into Eq. (70) and get

$$\begin{aligned} \{P_2, P_3\} = \frac{z_2^{2\mu-1} \sqrt{N_2 N_3} \zeta(1-\zeta)}{\mu_3} \Big\{ & f_{2a} [\zeta(1-\zeta)f''_{3b} + \mu_3(1-\mu)f_{3b}] \\ & + f_{3a} [\zeta(1-\zeta)f''_{2b} + \mu_3(1-\mu)f_{2b}] \Big\}. \end{aligned} \quad (73)$$

Thanks to the hypergeometric equation (29) applied to  $f_{3b}$  and  $f_{2b}$ , we see that the term under the square brackets cancels identically, and thus  $\{P_2, P_3\} = 0$ . Equations (69) eventually show that  $P_2$  and  $P_3$  are constants of motion:

$$\dot{P}_2 = 0, \quad \dot{P}_3 = 0. \quad (74)$$

### 4.3 Cartesian positions

We have just exhibited two constants of motion: The two relative Cartesian momenta of the particles. The Cartesian velocities, which are the time derivatives of the Cartesian positions  $Z_2$  and  $Z_3$ , should also be constant. We are going to define them as the variables conjugate to the momenta  $P_2$  and  $P_3$ .

The dilation factor  $D = p_2 z_2 + p_3 z_3$  may be expressed with the help of  $P_2$  and  $P_3$  in the form

$$D = z_2^{1-\mu} \left( \frac{\delta - \gamma\zeta}{\alpha\delta - \beta\gamma} P_2 - \frac{\beta - \alpha\zeta}{\alpha\delta - \beta\gamma} P_3 \right). \quad (75)$$

We define the new variables  $Z_2$  and  $Z_3$  in such a way that

$$D = (1 - \mu)(Z_2 P_2 + Z_3 P_3). \quad (76)$$

Let us work out the explicit expression for  $Z_2$  and  $Z_3$  as a function of the canonical variables. We introduce the notation

$$\mathcal{W} = \sqrt{N_2 N_3} (f_{2a} f_{3b} + f_{2b} f_{3a}), \quad (77)$$

which is the inner product of the spinors  $\sigma_a$  and  $\sigma_b$ , namely

$$\mathcal{W} = (\sigma_b, \sigma_a) \equiv \sigma_b^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma_a \quad (78)$$

and thus a  $SU(2)$  invariant. It turns out that  $\mathcal{W}$  has a simple expression, see Eq. (5.5) in Ref. [9]: As a matter of fact, it is a Wronskian function for the solutions of a second-order differential equation.

After comparison of Eqs. (75) and (76), the explicit expressions for  $Z_2$  and  $Z_3$  are

$$\begin{aligned} Z_2 &= z_2^{1-\mu} \frac{\sqrt{N_3} f_{3a}}{(1 - \mu) \mathcal{W}} \\ Z_3 &= z_2^{1-\mu} \frac{\sqrt{N_2} f_{2a}}{(1 - \mu) \mathcal{W}}. \end{aligned} \quad (79)$$

The pair  $Z \equiv (Z_2, -Z_3)$  makes up a spinor, in such a way that the spinorial product  $(P, Z) = P_2 Z_2 + P_3 Z_3$  be an invariant under  $SU(2)$  transformations.

In order to make contact with our intuition of classical physics, let us again take the small mass limit in which gravitational effects vanish. Then a straightforward calculation leads to

$$Z_3 = \sqrt{N_3} z_3, \quad Z_2 = \sqrt{N_2} \left( z_2 - \frac{\mu_3}{\mu_{13}} z_3 \right), \quad (80)$$

that is to say,  $Z_3$  is the position of particle #3 up to a normalization, and  $Z_2$  the one of particle #2 with respect to the system of particles (13). (We recall that we have chosen a frame in which  $z_1 = 0$ ).

The dilation factor  $D$  has a constant time derivative and the  $P_n$  are constants of motion, we expect that the  $Z_n$  also have constant time derivatives. This turns out to be true, and furthermore, the  $Z_n$ 's are canonically conjugate to  $P_n$ 's. The first point can be shown by evaluating

$$\dot{Z}_n = -\{H, Z_n\} = \frac{\{Z_n, P_2\} \bar{P}_2 + \{Z_n, P_3\} \bar{P}_3}{|P_2|^2 + |P_3|^2}. \quad (81)$$

Let us perform the calculation completely for  $n = 2$ . The Poisson bracket of  $Z_2$  and  $P_2$  that appears in Eq. (81) reads

$$\{Z_2, P_2\} = z_2^{\mu-1} [(1 - \mu)\alpha Z_2 + (\beta - \alpha\zeta)Z_2']. \quad (82)$$

To compute the derivative of  $Z$ , we use the more general formula

$$\frac{d}{d\zeta} \left( \frac{f'}{\mathcal{W}} \right) = -\frac{\mu_3(1-\mu)}{\zeta(1-\zeta)} \frac{f}{\mathcal{W}}, \quad (83)$$

valid for any solution  $f$  of the hypergeometric equation (29). We introduce the derivatives of  $f_{3b}$  in Eq. (79) expressed in Eq. (72). One then uses the previous formula with  $f$  set to  $f_{3b}$  to compute  $Z'_2$  in Eq. (82). With the help of the identities (71), one arrives at

$$\{Z_2, P_2\} = 1. \quad (84)$$

Similar calculations lead to the following Poisson brackets:

$$\{Z_2, P_3\} = 0, \quad \{Z_3, P_2\} = 0, \quad \{Z_3, P_3\} = 1, \quad \{Z_2, Z_3\} = 0, \quad \{P_2, P_3\} = 0. \quad (85)$$

Thus,

$$\dot{Z}_2 = \frac{\bar{P}_2}{|P_2|^2 + |P_3|^2}, \quad \dot{Z}_3 = \frac{\bar{P}_3}{|P_2|^2 + |P_3|^2}. \quad (86)$$

For consistency, we easily check that  $\dot{D} = 1 - \mu$ .

It is possible to recover the usual Cartesian equations of motion

$$\frac{dZ_2}{dT} = 2\bar{P}_2, \quad \frac{dZ_3}{dT} = 2\bar{P}_3 \quad (87)$$

by changing the time gauge as follows:

$$\frac{dt}{dT} = 2(|P_2|^2 + |P_3|^2). \quad (88)$$

This reparametrization depends on the variables  $z_n$  and  $p_n$ . With this choice for time and taking  $Z_n$  and  $P_n$  as canonical variables, the  $T$ -evolution would be given by the Hamiltonian

$$E = 2(|P_2|^2 + |P_3|^2), \quad (89)$$

up to a constant. Such a Hamiltonian would be simpler and more intuitive since it is the nonrelativistic kinetic energy of the three particles in the center-of-mass frame (see Eq. (41) and the discussion above it), but the phase-space variables would not be single valued.

## 5 Consistency with previous calculations and tentative extension to many bodies

In the present work, the computation of the Hamiltonian is based on a calculation of the total mass  $\mu = \sum_n \mu_n + \varepsilon$  as a function of the masses, positions and momenta of the particles in the framework of the second-order formalism.

On the other hand, since (2+1) gravity is a topological theory, we know that pointlike particles move on straight lines with constant velocities  $V_n$ , when appropriate (Cartesian) coordinates are chosen. The total mass  $\mu$  can be obtained by writing the total Cartesian momentum of the system of the 3 particles. The result for  $\mu$  should be the same as the one obtained in the previous section: This is what we are going to check here (Sec. 5.1). This computation will also help us to establish the relationship between  $V_n$  and the derivatives of  $Z_n$ . It can be extended to four (or more) particles and allow us to guess the form of the Hamiltonian in these cases (Sec. 5.2).

## 5.1 Comparison with the first-order formalism

Following Ref. [9], we introduce a spin- $\frac{1}{2}$  representation of the holonomy related to transport on a curve around the particle  $n$  of mass  $\mu_n$  and velocity  $V_n$  in the form

$$L_n(\mu_n, V_n) = \begin{pmatrix} \bar{a}_n & b_n \\ \bar{b}_n & a_n \end{pmatrix}, \quad \text{where} \quad \begin{cases} a_n = \cos \pi \mu_n + i \gamma_n \sin \pi \mu_n, \\ b_n = -i \gamma_n \bar{V}_n \sin \pi \mu_n, \end{cases} \quad (90)$$

and  $\gamma_n = 1/\sqrt{1-|V_n|^2}$ . In order to “measure” the total mass of the system, we may travel on a loop that goes around all the three particles. The total mass of the system  $\mu$  is then computed from the trace of the holonomy given by the product of the three matrices  $L_n$ :

$$\cos \pi \mu = \frac{1}{2} \text{Tr}[L_3(\mu_3, V_3) L_2(\mu_2, V_2) L_1(\mu_1, V_1)]. \quad (91)$$

The order of the product is determined by the choice of ordering the particles anticlockwise in space. The calculation leads to

$$\begin{aligned} \cos \pi \mu &= \cos \pi \mu_1 \cos \pi \mu_2 \cos \pi \mu_3 \\ &- \left( \gamma_1 \gamma_2 (1 - \vec{V}_1 \cdot \vec{V}_2) \sin \pi \mu_1 \sin \pi \mu_2 \cos \pi \mu_3 + [\text{cyclic permutations}] \right) \\ &+ \frac{i}{2} \gamma_1 \gamma_2 \gamma_3 (-V_{23} \bar{V}_1 + V_{13} \bar{V}_2 - V_{12} \bar{V}_3) \sin \pi \mu_1 \sin \pi \mu_2 \sin \pi \mu_3. \end{aligned} \quad (92)$$

We now expand to the lowest order in the velocities. We observe that the result can be written as the sum of the masses of the particles and of a quadratic form of the velocities, representing the total nonrelativistic kinetic energy of the system:

$$\mu = \mu_1 + \mu_2 + \mu_3 + \frac{1}{\pi} (\bar{V}_{12} \bar{V}_{13}) Q \begin{pmatrix} V_{12} \\ V_{13} \end{pmatrix}, \quad (93)$$

where

$$Q = \frac{1}{2s_{123}} \begin{pmatrix} s_2 s_{13} & -e^{-i\pi\mu_1} s_2 s_3 \\ -e^{i\pi\mu_1} s_2 s_3 & s_{12} s_3 \end{pmatrix}. \quad (94)$$

We may perform a standard decomposition of  $Q$  in a product of lower diagonal  $L$ , diagonal  $D$  and upper diagonal  $U$  matrices  $Q = LDU$ , with in this case,

$$D = \frac{1}{2} \begin{pmatrix} \frac{s_2 s_{13}}{s_{123}} & 0 \\ 0 & \frac{s_1 s_3}{s_{13}} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -e^{-i\pi\mu_1} \frac{s_3}{s_{13}} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L = U^\dagger \quad (95)$$

since  $Q$  is Hermitian. From this decomposition, we immediately write the kinetic energy as a sum of squared moduli:

$$\mu - \sum_n \mu_n = \frac{1}{2} \left( \frac{s_2 s_{13}}{\pi s_{123}} \left| \bar{V}_{12} - \frac{s_3 e^{i\pi\mu_1}}{s_{13}} \bar{V}_{13} \right|^2 + \frac{s_1 s_3}{\pi s_{13}} |V_{13}|^2 \right). \quad (96)$$

Note that the procedure used to arrive at this factorized expression may be repeated for a number of particles larger than 3. We sketch it in the next subsection 5.2.

A result of the first-order formalism developed in Ref. [9] is that the Cartesian velocities  $V_n$  can then be expressed with the help of the canonical positions  $z_n$  of the particles (see Eq. (4.19) in Ref. [9]). They read

$$\bar{V}_{1n} = 2K_0 z_2^{\mu-1} \int_0^{\zeta_n} d\xi \xi^{\mu_1-1} (\xi-1)^{\mu_2-1} (\xi-\zeta)^{\mu_3-1} (\xi-\eta_A), \quad (97)$$

where  $\eta_A$  is the apparent singularity. From the integral representation of the hypergeometric function

$$\int_0^1 dz z^{\alpha-1} (1-z)^{\beta-1} (1-tz)^{\gamma-1} \equiv B_{\alpha,\beta} F(\alpha, 1-\gamma, \alpha+\beta, t), \quad (98)$$

and after appropriate changes of variables, we express the  $\bar{V}$ 's in terms of the  $f$ 's defined in Eq. (28), namely

$$\begin{aligned} \bar{V}_{12} &= -2K_0 z_2^{\mu-1} e^{-i\pi\mu_2} \left\{ \frac{s_3}{s_{13}} e^{i\pi(\mu_1+\mu_3)} [f_{3b} + (\eta_A - \zeta)f_{3a}] + [f_{2b} - (\eta_A - \zeta)f_{2a}] \right\} \\ \bar{V}_{13} &= -2K_0 z_2^{\mu-1} e^{i\pi(\mu_3-\mu_2)} [f_{3b} + (\eta_A - \zeta)f_{3a}]. \end{aligned} \quad (99)$$

Inserting these expressions for the velocities in Eq. (96), we find

$$\mu - \sum_n \mu_n = 2|K_0|^2 |z_2|^{2\mu-2} \left( \frac{s_1 s_3}{\pi s_{13}} |(\eta_A - \zeta)f_{3a} + f_{3b}|^2 + \frac{s_2 s_{13}}{\pi s_{123}} |(\eta_A - \zeta)f_{2a} - f_{2b}|^2 \right) \quad (100)$$

Thus  $\varepsilon$  in Eq. (31) matches  $\mu - \sum \mu_i$  given by Eq. (100) from the first-order formalism provided

$$\eta_A = \zeta_A \quad \text{and} \quad |K_0|^2 = \frac{|K|^2}{8}. \quad (101)$$

Now we can establish the relation between the velocities, the Cartesian momenta and coordinates. Identifying Eq. (99) with the expressions of the momenta (35), we write

$$\begin{aligned} \bar{V}_{31} &= \frac{e^{i\pi(\mu_3-\mu_2)}}{\sqrt{2}} \frac{K}{D} P_3, \\ \bar{V}_{21} &= \frac{e^{i\pi(\mu_3-\mu_2)}}{\sqrt{2}} \frac{K}{D} \left( e^{-i\pi\mu_3} \frac{P_2}{\sqrt{N_2}} + e^{i\pi\mu_1} \frac{s_3}{s_{13}} \frac{P_3}{\sqrt{N_3}} \right). \end{aligned} \quad (102)$$

On the other hand, by definition of the Cartesian momenta  $P_2$  and  $P_3$ ,

$$\frac{|D|^2}{|K|^2} = \frac{|P_2|^2 + |P_3|^2}{4\varepsilon}. \quad (103)$$

The right-hand side is a numerical constant since  $\varepsilon$  has to be identified with the nonrelativistic kinetic energy in the Cartesian time gauge. Therefore

$$\begin{aligned} \bar{V}_{31} &= e^{i\phi} \frac{2P_3}{\sqrt{N_3}}, \\ \bar{V}_{21} &= e^{i\phi} \left( e^{-i\pi\mu_3} \frac{2P_2}{\sqrt{N_2}} + e^{i\pi\mu_1} \frac{s_3}{s_{13}} \frac{2P_3}{\sqrt{N_3}} \right), \end{aligned} \quad (104)$$

where we have introduced the angle  $\phi$  to absorb all irrelevant phases. One may now replace  $P_2$  and  $P_3$  by the derivative of  $Z_2$  and  $Z_3$  with respect to  $T$  (see Eq. (87)). Taking the small mass limit, with the help Eq. (80), it is easy to see that  $\bar{V}_{31}$  and  $\bar{V}_{21}$  coincide in this limit with the derivatives of  $z_3$  and  $z_2$  respectively.

## 5.2 Guessing the form of the Hamiltonian for many bodies

As was suggested before, we may express the total kinetic energy  $\varepsilon$  of a system of many bodies as a function of the Cartesian momenta of specific subsystems of particles. Then from a change of the time gauge, we may infer the form of the Hamiltonian. We address in some detail the case of the four-body system. We write a relation similar to Eq. (91):

$$\cos \pi\mu = \frac{1}{2} \text{Tr}[L_4(\mu_4, V_4) L_3(\mu_3, V_3) L_2(\mu_2, V_2) L_1(\mu_1, V_1)], \quad (105)$$



which we then expand at lowest order in the Cartesian velocities  $V_{1i}$

$$\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4 + \frac{1}{\pi} (\bar{V}_{12} \ \bar{V}_{13} \ \bar{V}_{14}) Q \begin{pmatrix} V_{12} \\ V_{13} \\ V_{14} \end{pmatrix}. \quad (106)$$

In this case, the matrix  $Q$  encoding the quadratic form of the velocities reads

$$Q = \frac{1}{2s_{1234}} \begin{pmatrix} s_2 s_{134} & -e^{-i\pi\mu_{14}} s_2 s_3 & -e^{-i\pi(\mu_1-\mu_3)} s_2 s_4 \\ -e^{i\pi\mu_{14}} s_2 s_3 & s_3 s_{124} & -e^{-i\pi\mu_{12}} s_3 s_4 \\ -e^{i\pi(\mu_1-\mu_3)} s_2 s_4 & -e^{i\pi\mu_{12}} s_3 s_4 & s_4 s_{123} \end{pmatrix}. \quad (107)$$

Performing as in the three-body case a  $Q = LDU$  decomposition we find the following formula for  $\varepsilon \equiv \mu - \sum_n \mu_n$ :

$$\varepsilon = \frac{1}{2} \left( \frac{s_2 s_{134}}{\pi s_{1234}} \left| \bar{V}_{12} - e^{i\pi(\mu_1+\mu_4)} \frac{s_3}{s_{134}} \bar{V}_{13} - e^{i\pi(\mu_1-\mu_3)} \frac{s_4}{s_{134}} \bar{V}_{14} \right|^2 + \frac{s_3 s_{14}}{\pi s_{134}} \left| \bar{V}_{13} - e^{i\pi\mu_1} \frac{s_4}{s_{14}} \bar{V}_{14} \right|^2 + \frac{s_1 s_4}{\pi s_{14}} |\bar{V}_{14}|^2 \right). \quad (108)$$

The identification with the classical equation for the total kinetic energy of the system (in the Cartesian time gauge)

$$\varepsilon = 2 (|P_2|^2 + |P_3|^2 + |P_4|^2) \quad (109)$$

leads (up to phases) to an expression of the Cartesian momenta  $P_4$ ,  $P_3$ ,  $P_2$  of particles number #4 relatively to #1, #3 relatively to the system (14), and #2 relatively to the system (134) respectively. This statement can be checked in the small mass limit in which the phases become irrelevant. We find for  $P_4$ ,  $P_3$  and  $P_2$  formulas similar to (39) and which, up to normalization factors, are the expressions of the momenta of free systems of particles as a function of their Cartesian velocities.

The strong similarity with the 3-body problem suggests that the Hamiltonian for  $N = 4$  particles and more generally, for an arbitrary number  $N$  of particles, would read

$$H = \ln \left( \frac{|P_2|^2 + |P_3|^2 + \dots + |P_N|^2}{4\varepsilon} \right). \quad (110)$$

Note however that the expressions of the  $P_n$  (or  $V_{1n}$ ) in the regular phase space coordinates  $(z_n, p_n)$  would involve line integrals in the complex plane with  $N$  cuts, which are not expected to be related in a simple way to known functions, as they are in the 2- and 3-body cases.

## 6 Towards the quantum Hilbert space

We impose canonical quantization by replacing the Poisson brackets  $\{\cdot, \cdot\}$  by the commutators  $-i\hbar[\cdot, \cdot]$ , which is realized by the substitution  $p_j \rightarrow -i\hbar\partial_{z_j}$ . In units  $\hbar = c = 1$ , this results in the usual relations between the positions  $z_i$  and momenta  $p_j$ , namely

$$[z_i, p_j] = i\delta_{ij}, \quad [z_i, z_j] = 0, \quad [p_i, p_j] = 0. \quad (111)$$

It turns out that the same rules apply to  $Z_i$  and  $P_j$ , provided the ordering of  $z_i$  and  $p_j$  is the one given in the defining equation (36) (i.e. the position operators are to the left of the momentum operators).

The problem then arises of finding an explicit expression for the eigenfunctions of the Hamiltonian, so as to construct the quantum Hilbert space and the scattering amplitudes. It turns out that it is not difficult to find explicit solutions for the Schrödinger equation, because the Hamiltonian is a simple function of the  $P_2$ ,  $P_3$  (and  $\bar{P}_2$ ,  $\bar{P}_3$ ) operators, which are diagonalized by the “plane waves”

$$\psi_{k_2, k_3}(z_2, z_3) = e^{i[k_2 Z_2(z_j) + k_3 Z_3(z_j) + (k_j \leftrightarrow \bar{k}_j, Z_j \leftrightarrow \bar{Z}_j)]}, \quad (112)$$

where  $k_2$ ,  $k_3$  label the eigenvalues of  $P_2$ ,  $P_3$  and  $Z_2$ ,  $Z_3$  are expressed in terms of the regular coordinates  $z_j$  by Eq. (79). The corresponding energy eigenvalue of Eq. (42) is  $E = 2(|k_2|^2 + |k_3|^2)$ .

However, the wave functions (112) are not single valued, but transform in a way induced by the monodromy transformations  $M$  of Sec. 3.3. In fact, if we let  $z_2 \rightarrow z_2^M = z_2 e^{2i\pi}$  or  $z_3 \rightarrow z_3^M = z_3 e^{2i\pi}$  or combinations thereof, then  $(Z_2, -Z_3)$  transforms as a spinor. The invariant combination in the exponent of (112) translates this transformation into a  $U(2)$  transformation of the momenta:  $k \equiv (k_3, k_2) \rightarrow k^M = (k_3^M, k_2^M)$ . The transformed wave function reads

$$\psi_k(z_j^M) = e^{i[k_2^M Z_2(z_j) + k_3^M Z_3(z_j) + \text{c.c.}]}. \quad (113)$$

This wave function possesses the same energy eigenvalue  $E = 2(|k_2|^2 + |k_3|^2)$  as  $\psi_k(z_j)$  in Eq. (112). In other words,  $\psi_k(z_j)$  transforms into another wave function of the same degeneracy class according to a  $U(2)$  representation of the monodromy.

A similar situation arises already at the level of the simple two-body case, where the analog of Eq. (112) is the wave function

$$\psi_k(z) = e^{i[kZ(z) + \text{c.c.}]} = e^{i[kz^{1-\mu}/(1-\mu) + \text{c.c.}]}, \quad E = 2|k|^2. \quad (114)$$

Similarly, it is not monodromic, but transforms by a  $U(1)$  monodromy as

$$\psi_k(z) \rightarrow \psi_k(e^{2i\pi} z) = \psi_{ke^{2i\pi\alpha}}(z), \quad (\alpha = 1 - \mu), \quad (115)$$

which remains in the same degeneracy class. One may think that in the two-body  $U(1)$  case, it would be possible to construct linear combinations of plane waves which are single valued and have the same energy by just summing with unit weight over the whole monodromy group, as follows:

$$\tilde{\psi}_k(z) = \sum_{n=-\infty}^{+\infty} \psi_k(e^{2i\pi n} z). \quad (116)$$

However, this turns out to be too naive. Unless the monodromy group is finite (for fractional values of  $\alpha$ ), the series sums up to zero, after appropriate regularization.

A solution which is both finite and monodromic was proposed by Deser, Jackiw [7] and 't Hooft [6]. It also consists in writing a superposition of plane waves degenerate in energy (which differ by the azimuthal angle of their momenta), but with nontrivial weights. Let us introduce some useful notations:

$$x \equiv |k||z|^\alpha/\alpha, \quad \beta \equiv \arg(k), \quad \theta \equiv -\arg z. \quad (117)$$

Then

$$\psi_k(z) = \int_{\mathcal{C}_{\alpha\theta}} \frac{d\beta}{2\pi} \hat{f}(\beta) e^{ix \cos(\beta - \alpha\theta)} \quad (118)$$

solves the Schrödinger equation when the contour  $\mathcal{C}_{\alpha\theta}$  is a combination of Schläfli contours in the upper/lower complex plane of the  $\beta$  variable, namely the union of the broken line

$$\mathcal{C}_+ \equiv ] -\pi + i\infty, -\pi, \pi, \pi + i\infty[ \quad (119)$$

translated along the real axis by  $\alpha\theta$  and of its complex conjugate  $\mathcal{C}_-$  (we introduce the notation  $\mathcal{C} \equiv \mathcal{C}_+ + \mathcal{C}_-$  for the union), with appropriate deformations in order to avoid the possible singularities of  $\hat{f}$ . The function  $\hat{f}$  is meromorphic and well behaved for  $\text{Im}(\beta) \rightarrow \pm\infty$ . The convergence lines in  $\beta$  are  $\beta - \alpha\theta \rightarrow (2n+1)\pi \pm i\infty$ , thus the contour  $\mathcal{C}_{\alpha\theta}$  may be translated by  $\beta \rightarrow \beta \pm 2\pi n$ .

For each choice of  $\mathcal{C}$  one may now construct a monodromic solution by replacing  $\theta$  by  $\theta_n \equiv \theta + 2n\pi$  and summing over  $n$ :

$$\tilde{\psi}_k(z) = \sum_n \int_{\mathcal{C}_{\alpha\theta_n}} \frac{d\beta}{2\pi} \hat{f}(\beta) e^{ix \cos(\beta - \alpha\theta_n)} = \int_{\mathcal{C}} \frac{d\beta}{2\pi} \left[ \sum_n \hat{f}(\beta + \alpha\theta_n) \right] e^{ix \cos \beta}. \quad (120)$$

The function

$$f(\beta) \equiv \sum_n \hat{f}(\beta + 2\pi\alpha n) \quad (121)$$

which appears under the square brackets is periodic of period  $2\pi\alpha$ , so that  $\psi_k$  is monodromic under the rotation  $\theta \rightarrow \theta + 2\pi$ .

Such a monodromic expression can then be projected on the integer angular momentum  $m$  as follows:

$$\psi_k^m(z) = \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} e^{-im\theta} \int_{\mathcal{C}} \frac{d\beta}{2\pi} \hat{f}(\beta + \alpha\theta) e^{ix \cos \beta} = \int_{\mathcal{C}} \frac{d\beta}{2\pi} \left[ \int_{-\pi}^{+\pi} \frac{d\theta}{2\pi} f(\beta + \alpha\theta) e^{-im\theta} \right] e^{ix \cos \beta}. \quad (122)$$

The integral over  $\theta$  (under the square brackets) yields the Fourier coefficient  $f_m$  of the periodic function  $f$  times the phase  $e^{im\beta/\alpha}$ . The integral over  $\beta$  is then seen to yield the Bessel function  $J_{|m|/\alpha}(x)$ , which is the DJH [7, 6] result. The rôle of the weight factor  $\hat{f}$  is to suppress the contributions of the secondary sheets so as to avoid too much destructive interference. For the DJH scattering process  $\hat{f} \sim 1/(\beta - \pi\alpha)$  and  $f \sim \tan(\beta/2\alpha)$ .

Is a similar procedure available for  $U(2)$  (or more generally  $U(N-1)$ )? On one hand, when the monodromy group is finite, we may write

$$\tilde{\psi}_k(z_j) = \sum_{\text{monodromies } M} \psi_{kM}(Z(z_j)) = \sum_M e^{i[k_2^M Z_2(z_j) + k_3^M Z_3(z_j) + \text{c.c.}]}, \quad (123)$$

where  $k$  is now a  $U(2)$  spinor with components  $(k_3, k_2)$ , and  $Z$  has components  $(Z_2, -Z_3)$ . But on the other hand, despite various attempts, we have been unable to extend to this non-abelian case the general harmonic analysis of Eq. (122). The main obstacle is the nonabelian group multiplicative structure which does not allow an explicit evaluation (with identification of good quantum numbers).

Let us however give an example that shows how a simple structure may arise for proper (rational) values of the masses. Let us consider the case in which the particle masses are  $\mu_1 = \mu_2 = \mu_3 = \frac{1}{4}$ . Then the three basic monodromy matrices are related to the Pauli matrices  $\sigma_1, \sigma_2$  and  $\sigma_3$  through

$$M_{31} = -\sigma_3, \quad M_{32} = \sigma_1, \quad M_{21} = \sigma_2. \quad (124)$$

The group generated by the monodromies is finite in this case. It is made of the sixteen  $U(2)$  matrices  $\epsilon I, \epsilon\sigma_1, \epsilon\sigma_2, \epsilon\sigma_3$ , where  $\epsilon \in \{1, i, -1, -i\}$ .

In this particular case, the Cartesian coordinates have a simple expression as a function of the regular coordinates  $z_2$  and  $\zeta = z_3/z_2$ . Equations (79) indeed boil down to

$$Z_2 = \frac{4\pi}{\Gamma^2(\frac{1}{4})} z_2^{1/4} \left(1 + \sqrt{1 - \zeta}\right)^{1/2}, \quad Z_3 = \frac{4\pi}{\Gamma^2(\frac{1}{4})} z_2^{1/4} \left(1 - \sqrt{1 - \zeta}\right)^{1/2}. \quad (125)$$

We now apply Eq. (123) to obtain the explicit monodromic wave function:

$$\begin{aligned} \tilde{\psi}_k(z_2, \zeta) = \sum_{\epsilon \in \{1, i, -1, -i\}} & \left\{ e^{i[\epsilon k_2 z_2^{1/4} (1 + \sqrt{1 - \zeta})^{1/2} + \epsilon k_3 z_2^{1/4} (1 - \sqrt{1 - \zeta})^{1/2} + \text{c.c.}]} \right. \\ & + e^{i[\epsilon k_2 z_2^{1/4} (1 - \sqrt{1 - \zeta})^{1/2} + \epsilon k_3 z_2^{1/4} (1 + \sqrt{1 - \zeta})^{1/2} + \text{c.c.}]} \\ & + e^{i[\epsilon k_2 z_2^{1/4} (1 + \sqrt{1 - \zeta})^{1/2} - \epsilon k_3 z_2^{1/4} (1 - \sqrt{1 - \zeta})^{1/2} + \text{c.c.}]} \\ & \left. + e^{i[-i \epsilon k_2 z_2^{1/4} (1 - \sqrt{1 - \zeta})^{1/2} + i \epsilon k_3 z_2^{1/4} (1 + \sqrt{1 - \zeta})^{1/2} + \text{c.c.}]} \right\}. \quad (126) \end{aligned}$$

## 7 Summary and suggestions

In this paper, we have fully understood the canonical Hamiltonian structure of the 3-body problem in the nonrelativistic limit in which the particle velocities are small. Let us summarize our findings, repeating the explicit formulas to which we have arrived.

We have provided the expression of the Hamiltonian in the form

$$H = \frac{1}{2\kappa^2} \ln \frac{(2\kappa^2)^2 (|P_2|^2 + |P_3|^2)}{4\varepsilon} \quad (127)$$

where  $P_2$  and  $P_3$  are properly-defined relative momenta, which are given as explicit functions of the canonical coordinates  $(z_2, p_2)$  and  $(z_3, p_3)$ , while  $\varepsilon$  is a fixed parameter which does not play a rôle in the dynamics: It represents the difference between the total (adimensionalized) mass of the Universe  $\mu$ , and the sum of the masses  $\mu_n$  of the particles, which are all constants. The nonrelativistic limit implies that it is small compared to  $\mu$ .

By putting Eqs. (36) and (37) together, we obtain the expressions

$$\begin{aligned} P_3 &= z_2^\mu \sqrt{N_3} [f_{3b}(p_2 + \zeta p_3) + f_{3a}\zeta(1 - \zeta)p_3], \\ P_2 &= z_2^\mu \sqrt{N_2} [f_{2b}(p_2 + \zeta p_3) - f_{2a}\zeta(1 - \zeta)p_3], \end{aligned} \quad (128)$$

where the normalization factors  $N_3$  and  $N_2$  are given by Eq. (32), namely

$$N_3 = \frac{\sin \pi \mu_1 \sin \pi \mu_3}{\pi \sin \pi \mu_{13}}, \quad N_2 = \frac{\sin \pi \mu_2 \sin \pi \mu_{13}}{\pi \sin \pi \mu}, \quad (129)$$

where  $\mu_{ij} = \mu_i + \mu_j$  and  $\mu = \mu_1 + \mu_2 + \mu_3$ . The  $f$ 's are solutions of the hypergeometric equation with specific coefficients (see Eq. (29)), and their expressions read (28)

$$\begin{aligned} f_{3a} &= \frac{\Gamma(\mu_1)\Gamma(\mu_3)}{\Gamma(\mu_{13})} \zeta^{\mu_{13}-1} F(\mu_1, 1 - \mu_2, \mu_{13}, \zeta), \\ f_{2a} &= \frac{1 - \mu}{1 - \mu_{13}} \frac{\Gamma(\mu_{13})\Gamma(\mu_2)}{\Gamma(\mu)} F(1 - \mu_3, 2 - \mu, 2 - \mu_{13}, \zeta), \end{aligned} \quad (130)$$

and  $f_{3b}$ ,  $f_{2b}$  are obtained from  $f_{3a}$ ,  $f_{2a}$  respectively by shifting  $\mu_3$  to  $\mu_3 + 1$ .  $P_3$  and  $P_2$  are constants of motion which are related to the Cartesian momenta.

One can also build the Cartesian coordinates, which have relatively simple expressions. From Eq. (79), we get

$$\begin{aligned} Z_2 &= \frac{z_2^{1-\mu}}{1 - \mu} \frac{\Gamma(\mu)}{\sqrt{N_2}\Gamma(\mu_{13})\Gamma(\mu_2)} F(\mu_3, \mu - 1, \mu_{13}, \zeta), \\ Z_3 &= \frac{z_2^{1-\mu}}{1 - \mu_{13}} \frac{\Gamma(\mu_{13})}{\sqrt{N_3}\Gamma(\mu_1)\Gamma(\mu_3)} \zeta^{1-\mu_{13}} F(\mu_2, 1 - \mu_1, 2 - \mu_{13}, \zeta). \end{aligned} \quad (131)$$

Finally, the Cartesian time  $T$  is related to the ADM time  $t$  by

$$\frac{dT}{dt} = \frac{1}{2(2\kappa^2)^2(|P_2|^2 + |P_3|^2)}, \quad (132)$$

which represents a time-gauge change.

An important rôle is played by the  $U(2)$  symmetry of  $H$ , under which  $(P_3, P_2)$  and  $(Z_2, -Z_3)$  transform as spinors. This symmetry regulates the monodromies and in particular implies that  $H$  is invariant, that is, it is single valued. It is also useful to represent the exchange symmetries of the problem, e.g. the  $2 \leftrightarrow 3$  exchange, under which the Hamiltonian is also invariant. We have argued that the  $U(2)$  symmetry will be replaced by a  $U(N-1)$  symmetry in the nonrelativistic  $N$ -body case.

Given our understanding of the Hamiltonian structure, canonical quantization is in principle straightforward, but the construction of the canonical Hilbert space is not. We have seen that in the 2-body case, it is possible to find monodromic eigenfunctions by projecting over the  $U(1)$  gauge variable  $\theta$  from  $-\infty$  to  $+\infty$ , after a weighted sum over monodromies and careful regularization. It is also possible that a similar sum and projection (for instance a harmonic projection on  $U(2)$  or on a proper subgroup) is able to define monodromic wave functions for the 3-body system, together with their relevant quantum numbers. Further analysis is needed in this direction.

**Acknowledgments** We acknowledge useful discussions with A. Cappelli and D. Seminara. The work of SM is partly supported by the Agence Nationale de la Recherche (France), contract ANR-06-JCJC-0084-02. He thanks the Department of Physics of the University of Florence and the INFN for hospitality and support at various stages of this work.

## A $SU(2)$ monodromies and relabeling symmetry

### A.1 Monodromies

We compute the monodromies of the spinors  $\sigma_a$  and  $\sigma_b$  in Eq. (43) and we check that they indeed transform according to the matrices  $M_{31}$  and  $M_{32}$  in Eqs. (45) and (46) when the particles loop around each other. While the transformation  $\zeta \rightarrow e^{2i\pi}\zeta$  is straightforward to perform on the expressions of  $f_{3b}$  and  $f_{2b}$  given in Eq. (28) and leads to the diagonal matrix  $M_{31}$  (see Eq. (45)), the monodromy around the particle at position  $\zeta = 1$  is a bit more tricky. The details being quite lengthy, we provide here only the main steps in order for the reader to be able to reproduce the full calculation.

We start from the expressions of  $f_{3b}$  and  $f_{2b}$  in Eq. (28), and we apply the well-known hypergeometric transformation [19]

$$F(a, b, c, z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, a+b-c+1, 1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a, c-b, 1+c-a-b, 1-z) \quad (133)$$

in order to change the argument of the latter from  $\zeta$  to  $1-\zeta$ . Then, since the obtained hypergeometric functions are analytic around the point  $1-\zeta = 0$ , the monodromy transformations may be read from the prefactors. We find

$$f_{3b} = \zeta^{\mu_1+\mu_3} \frac{\Gamma(\mu_1+\mu_2)\Gamma(\mu_1)}{\Gamma(\mu)} F(\mu_1, 1-\mu_2, 1-\mu_2-\mu_3, 1-\zeta) + \zeta^{\mu_1+\mu_3} (1-\zeta)^{\mu_2+\mu_3} \frac{\Gamma(1+\mu_3)\Gamma(-\mu_2-\mu_3)}{\Gamma(1-\mu_2)} F(1+\mu_3, \mu, 1+\mu_2+\mu_3, 1-\zeta). \quad (134)$$

As for  $f_{2b}$ , a similar transformation may be applied, which leads to

$$f_{2b} = \frac{\Gamma(\mu_1+\mu_3)\Gamma(1-\mu_1-\mu_3)\Gamma(\mu_2+\mu_3)}{\Gamma(\mu)\Gamma(1-\mu_1)} F(-\mu_3, 1-\mu, 1-\mu_2-\mu_3, 1-\zeta) + \frac{\Gamma(\mu_1+\mu_3)\Gamma(\mu_2)\Gamma(1-\mu_1-\mu_3)\Gamma(-\mu_2-\mu_3)}{\Gamma(-\mu_3)\Gamma(1-\mu)\Gamma(\mu)} (1-\zeta)^{\mu_2+\mu_3} F(1-\mu_1, \mu_2, 1+\mu_2+\mu_3, 1-\zeta). \quad (135)$$

Then, one applies a further transformation to the two hypergeometric functions which appear in the previous expression,

$$F(-\mu_3, 1-\mu, 1-\mu_2-\mu_3, 1-\zeta) = \zeta^{\mu_1+\mu_3} F(1-\mu_2, \mu_1, 1-\mu_2-\mu_3, 1-\zeta) \\ F(1-\mu_1, \mu_2, 1+\mu_2+\mu_3, 1-\zeta) = \zeta^{\mu_1+\mu_3} F(\mu, 1+\mu_3, 1+\mu_2+\mu_3, 1-\zeta) \quad (136)$$

which enables one to write  $f_{2b}$  in a form that is similar to  $f_{3b}$  as far as the hypergeometric functions are concerned:

$$f_{2b} = \frac{s_1}{s_{13}} \frac{\Gamma(\mu_2+\mu_3)\Gamma(\mu_1)}{\Gamma(\mu)} \zeta^{\mu_1+\mu_3} F(\mu_1, 1-\mu_2, 1-\mu_2-\mu_3, 1-\zeta) - \frac{s_{123}s_3}{s_{13}s_2} \frac{\Gamma(1+\mu_3)\Gamma(-\mu_2-\mu_3)}{\Gamma(1-\mu_2)} \zeta^{\mu_1+\mu_3} (1-\zeta)^{\mu_2+\mu_3} F(1+\mu_3, \mu, 1+\mu_2+\mu_3, 1-\zeta). \quad (137)$$

One now recognizes that the hypergeometric functions which appear in the transformed expressions for the  $f_{3b}$  and  $f_{2b}$  are the same.

In this form, the hypergeometric functions are invariant by the monodromy transformation  $\zeta - 1 \rightarrow e^{2i\pi}(\zeta - 1)$ . Only the prefactors transform, and in a trivial way. Lengthy but straightforward calculations lead to an expression of the relationship between the transformed of the vector  $(f_{3b}, f_{2b})$  and its untransformed form through the multiplication by a matrix. Setting the relative normalizations in front of  $f_{3b}$  and  $f_{2b}$  as in Eq. (43), we find that this matrix is precisely  $M_{32}$  given in Eq. (45). Thus  $\sigma_b$  transforms as a  $SU(2)$  spinor (up to a  $U(1)$  phase) under monodromy transformations.

## A.2 Label exchange symmetry

We start from Eq. (44) for the momentum  $P = (P_3, P_2)$ , namely

$$P = Dz_2^{\mu-1} [\sigma_b + (\zeta_A - \zeta)\sigma_a]. \quad (138)$$

The dilation factor  $D$  is obviously invariant under any relabeling, see its definition in Eq. (7).

Let us first exchange the labels of particles #1 and #3. This amounts to replacing  $p_3$  by  $p_1 = -p_2 - p_3$ , to performing the conformal transformation  $\zeta \rightarrow \zeta/(\zeta - 1)$  and the substitution  $z_2 \rightarrow z_2(1 - \zeta)$ . The transformed momenta read

$$\tilde{P} = Dz_2^{\mu-1}(1 - \zeta)^{\mu-1} \left( \tilde{\sigma}_b + \frac{\zeta_A}{1 - \zeta} \tilde{\sigma}_a \right), \quad (139)$$

where we have put a “tilde” sign above the transformed quantities.

We need to understand how  $\sigma_b$  and  $\sigma_a$  transform. The normalization factors  $\sqrt{N_2}$  and  $\sqrt{N_3}$  are invariant in this case, as seen from their definitions (32). As for the transformations of the various functions  $f$  defined in Eq. (28), the key formula is the following identity between hypergeometric functions:

$$F(a, b, c, \zeta/(\zeta - 1)) = (1 - \zeta)^b F(c - a, b, c, \zeta). \quad (140)$$

Using this formula, expressing  $\tilde{\sigma}_a$  is then straightforward. Defining

$$\tau_{31} \equiv \begin{pmatrix} -e^{i\pi\mu_{13}} & 0 \\ 0 & 1 \end{pmatrix}, \quad (141)$$

we obtain

$$\tilde{\sigma}_a = (1 - \zeta)^{2-\mu} \tau_{31} \cdot \sigma_a. \quad (142)$$

As for the transformation of the components  $f_{3b}$  and  $f_{2b}$  of  $\sigma_b$ , one needs to use of the respective contiguity relations for hypergeometric functions

$$(c - 1)F(a, b, c - 1, z) - bF(a, b + 1, c, z) + (b - c + 1)F(a, b, c, z) = 0 \quad (143)$$

with  $a = 1 - \mu_2$ ,  $b = \mu_1$ ,  $c = 1 + \mu_{13}$ ,  $z = \zeta$ , and

$$cF(a, b, c, z) - bzF(a, b + 1, c + 1, z) - cF(a - 1, b, c, z) = 0 \quad (144)$$

with  $a = 1 - \mu_3$ ,  $b = 1 - \mu$ ,  $c = 1 - \mu_{13}$ ,  $z = \zeta$ . We arrive at

$$\tilde{\sigma}_b = (1 - \zeta)^{1-\mu} \tau_{31} \cdot (\sigma_b - \zeta \sigma_a). \quad (145)$$

Combining Eqs. (142) and (145) with Eq. (139), one easily finds

$$\tilde{P} = \tau_{31} \cdot P. \quad (146)$$

We recognize that  $\tau_{31}$  is the transformation matrix (49). (Note that the phases stem from the replacement of  $\zeta - 1$  by  $e^{-i\pi}(1 - \zeta)$ . The convention with the opposite sign for the phase would have led to a different transformation matrix).

Next, we exchange the labels of particles #2 and #3. This amounts to exchanging  $p_2$  and  $p_3$ , to transforming  $\zeta$  into  $1/\zeta$  and to substituting  $z_2$  by  $z_2\zeta$ . The transformed momenta thus read

$$\tilde{P} = Dz_2^{\mu-1}\zeta^{\mu-1} \left( \tilde{\sigma}_b + \frac{\zeta_A - 1}{\zeta} \tilde{\sigma}_a \right). \quad (147)$$

The transformations of  $\sqrt{N_2}$  and  $\sqrt{N_3}$  are straightforward from their definitions (32). Again, expressing the various components  $\tilde{f}$  of  $\tilde{\sigma}$  with the help of the components of  $\sigma$  requires involved manipulations of the hypergeometric functions. We first need to use the formula

$$\begin{aligned} F(a, b, c, 1/z) &= \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} (e^{-i\pi}z)^a F(a, a-c+1, a-b+1, z) \\ &\quad + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} (e^{-i\pi}z)^b F(b, b-c+1, b-a+1, z) \end{aligned} \quad (148)$$

in order to express the  $\tilde{f}$ 's, which are hypergeometric functions of argument  $1/\zeta$ , as linear combinations of the  $f$ 's. Without any further transformation, we arrive at

$$\tilde{\sigma}_a = \zeta^{2-\mu} \tau_{32} \cdot \sigma_a, \quad (149)$$

with

$$\tau_{32} \equiv \frac{1}{\sqrt{s_{12}s_{13}}} \begin{pmatrix} e^{-i\pi\mu_1} \sqrt{s_2 s_3} & e^{i\pi\mu_3} \sqrt{s_1 s_{123}} \\ e^{i\pi\mu_2} \sqrt{s_1 s_{123}} & -e^{i\pi\mu} \sqrt{s_2 s_3} \end{pmatrix}. \quad (150)$$

The functions  $\tilde{f}_{3b}$ ,  $\tilde{f}_{2b}$  are deduced from the functions  $\tilde{f}_{3a}$ ,  $\tilde{f}_{2a}$  by simply replacing  $\mu_2$  by  $\mu_2 + 1$ . The hypergeometric functions  $F(-\mu_2, \mu_1, \mu_{13}, \zeta)$  and  $F(1 - \mu, 1 - \mu_3, 2 - \mu_{13}, \zeta)$  then appear, which are not part of our original basis  $f_3, f_2$ . But they can actually be expressed as linear combinations of the  $f$ -functions using contiguity relations. Thanks to the identity

$$cF(a, b, c, z) + (a - c)zF(a, b + 1, c + 1, z) + (z - 1)cF(a, b + 1, c) = 0 \quad (151)$$

with  $a = \mu_1$ ,  $b = -\mu_2$ ,  $c = \mu_{13}$ ,  $z = \zeta$ , we write

$$F(\mu_1, -\mu_2, \mu_{13}, z) = \frac{\Gamma(\mu_{13})}{\Gamma(\mu_1)\Gamma(\mu_3)} \zeta^{1-\mu_{13}} [f_{3b} + (1 - \zeta)f_{3a}]. \quad (152)$$

Using

$$\begin{aligned} (a - b)(a - c + 1)F(a, b + 1, c, z) + a(a - b)(z - 1)F(a + 1, b + 1, c, z) \\ + (c - 1)(a - b)F(a, b, c - 1, z) = 0 \end{aligned} \quad (153)$$

with  $a = 1 - \mu$ ,  $b = -\mu_3$ ,  $c = 2 - \mu_{13}$ ,  $z = \zeta$ , we get

$$F(1 - \mu, 1 - \mu_3, 2 - \mu_{13}, \zeta) = -\frac{\Gamma(\mu)}{\Gamma(1 + \mu_2)\Gamma(\mu_{13} - 1)} [f_{2b} - (1 - \zeta)f_{2a}]. \quad (154)$$

Inserting these identities in  $\tilde{\sigma}_b$  expressed with the help of the hypergeometric functions of argument  $\zeta$ , we find

$$\tilde{\sigma}_b = \zeta^{1-\mu} \tau_{32} \cdot [\sigma_b + (1 - \zeta)\sigma_a]. \quad (155)$$

The replacement of  $\tilde{\sigma}_a$  and  $\tilde{\sigma}_b$  expressed with the help of  $\sigma_a$  and  $\sigma_b$  (Eq. (149) and (155)) into Eq. (147) leads to the transformation

$$\tilde{P} = \tau_{32} \cdot P, \quad (156)$$

which is an other way to write Eq. (50).



## References

- [1] A. Staruszkiewicz, *Acta Phys. Polon.* **24**, 735 (1963).
- [2] S. Carlip, “Quantum gravity in 2+1 dimensions,” Cambridge University Press (1998).
- [3] S. Carlip, *Living Rev. Rel.* **8**, 1 (2005).
- [4] S. Carlip, S. Deser, A. Waldron and D. K. Wise, *Phys. Lett. B* **666**, 272 (2008).
- [5] S. Deser, R. Jackiw, G. ’t Hooft, *Annals Phys.* **152**, 220 (1984).
- [6] G. ’t Hooft, *Commun. Math. Phys.* **117**, 685 (1988).
- [7] S. Deser and R. Jackiw, *Commun. Math. Phys.* **118**, 495 (1988).
- [8] A. Bellini, M. Ciafaloni and P. Valtancoli, *Nucl. Phys. B* **454** (1995) 449.
- [9] A. Bellini, M. Ciafaloni and P. Valtancoli, *Nucl. Phys. B* **462** (1996) 453.
- [10] A. Bellini and P. Valtancoli, *Phys. Lett. B* **348**, 44 (1995).
- [11] L. Cantini, P. Menotti and D. Seminara, *Class. Quant. Grav.* **18** (2001) 2253.
- [12] M. Yoshida, “Fuchsian differential equations,” *Aspects of Mathematics*, E11, Friedr. Vieweg & Sohn, Braunschweig, 1987.
- [13] P. Menotti and D. Seminara, *Annals Phys.* **279** (2000) 282.
- [14] M. Welling, *Class. Quant. Grav.* **13** (1996) 653.
- [15] R. L. Arnowitt, S. Deser and C. W. Misner, *Phys. Rev.* **116**, 1322 (1959).
- [16] R. L. Arnowitt, S. Deser and C. W. Misner, “Gravitation: An Introduction to Current Research”, edited by L. Witten (Wiley, 1962).
- [17] S. W. Hawking and C. J. Hunter, *Class. Quant. Grav.* **13** (1996) 2735.
- [18] For a recent perturbative approach to the calculation of the Hamiltonian, see A. Yale, R. B. Mann and T. Ohta, *Class. Quant. Grav.* **27**, 245005 (2010).
- [19] I. S. Gradshteyn and I. M. Ryzhik, in “Tables of Integrals, Series and Products,” edited by Alan Jeffrey (Academic, New York, 1965).
- [20] V. S. Dotsenko and V. A. Fateev, *Nucl. Phys. B* **240**, 312 (1984).
- [21] J. S. Geronimo and H. Navelet, *J. Math. Phys.* **44**, 2293 (2003).
- [22] E. Artin, *Ann. Math.* **48**, 101-126 (1947).